Indian Statistical Institute, Bangalore
M. Math.
First Year, Second Semester
Functional Analysis

Mid-term Examination | Date: February 25, 2020
Maximum marks: 100 | Time: 3 hours
Instructor: B V Rajarama Bhat

(1) Show that any two norms on a finite dimensional normed linear space over $\mathbb{R}$ are equivalent.
[15]

(2) Let $(X, \|\cdot\|)$ be a normed linear space. Let the unit sphere of X be defined by:
$$S_1(X) = \{x \in X : \|x\| = 1\}.$$
Let D be a convex subset of $S_1(X)$ and $u, v \in D$ with $u \neq v$. (i) Show that $\|u+v\| = \|u\| + \|v\|$. (ii) Show that u, v are linearly independent. (iii) Give an example of a pair of vectors u, v in some normed linear space X as above. [15]

(3) Let $C_{\mathbb{R}}[0,1]$ be the Banach space of real valued continuous functions on $[0,1]$ with
$$\|f\| = \sup\{|f(t)| : t \in [0,1]\}$$
for $f \in C_{\mathbb{R}}[0,1]$. Define $T : C_{\mathbb{R}}[0,1] \to C_{\mathbb{R}}[0,1]$ by
$$Tf(t) = t\int_0^t f(s)ds \ \forall f \in C_{\mathbb{R}}[0,1].$$
(i) Show that T is a bounded linear map. (ii) Show that T is $1-1$ but not onto.
[15]

(4) Let $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ be Cauchy sequences in a normed linear space X. Show that $\{\|x_n - y_n\|\}_{n\geq 1}$ is a convergent sequence. Give an example of this kind where $\{x_n - y_n\}_{n\geq 1}$ is not convergent. [15]

(5) Let $(X, \|\cdot\|)$ be a normed linear space over $\mathbb{R}$ and let $f : X \to \mathbb{R}$ be a non-zero bounded linear functional. Define $N = \{x \in X : f(x) = 0\}$. For $x \in X$, define
$$d(x, N) := \inf\{\|x - y\| : y \in N\}.$$
Show that
$$d(x, N) = \frac{|f(x)|}{\|f\|}.$$
[15]

(6) Let l^2 be the Hilbert space of square summable sequences of complex numbers with standard ortho-normal basis $\{e_n : n \in \mathbb{N}\}$. Define $Te_1 = e_2$ and $Te_n = e_{n-1} + e_{n+1}$, $\forall n \geq 2$. (i) Show that T extends to a bounded linear operator on l^2. (ii) Compute T^*. (iii) Show that 1 is not an eigenvalue of T. [15]

(7) Let Y be a closed subspace of a Banach space X. The *annihilator* of Y, denoted by Y^a, is defined as
$$Y^a := \{f \in X' : f(y) = 0 \ \forall \ y \in Y\},$$
where X' is the dual space of X. (i) Show that Y^a is a closed subspace of X' and $(Y^a)^a$ is a closed subspace of X''. (ii) Suppose $i : X \to X''$ is the canonical map. Show that $i(Y) \subseteq (Y^a)^a$. (iii) Show that if X is reflexive then $i(Y) = (Y^a)^a$.
[15]